CONGRUENCE B-ORBITS AND THE BRUHAT POSET OF INVOLUTIONS OF THE SYMMETRIC GROUP

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ABSTRACT. We study the poset of Borel congruence classes of symmetric matrices ordered by containment of closures. We give a combinatorial description of this poset and calculate its rank function. We discuss the relation between this poset and the Bruhat poset of involutions of the symmetric group.

1. INTRODUCTION

A remarkable property of the Bruhat decomposition of $GL_n(\mathbb{C})$ (i.e. the decomposition of $GL_n(\mathbb{C})$ into double cosets $\{B_1\pi B_2\}$ where $\pi \in S_n$, $B_1, B_2 \in \mathbb{B}_n(\mathbb{C})$ – the subgroup of upper-triangular invertible matrices) is that the natural order on double cosets (defined by containment of closures) leads to the same poset as the combinatorially defined Bruhat order on permutations of S_n (for $\pi, \sigma \in S_n$, $\pi \leqslant \sigma$ if π is a subword of σ with respect to the reduced form in Coxeter generators). L. Renner introduced and developed the beautiful theory of Bruhat decomposition for not necessarily invertible matrices, see [8] and [7]. When the Borel group acts on all the matrices, the double cosets are in bijection with partial permutations which form a so called rook monoid R_n which is the finite monoid whose elements are the 0-1 matrices with at most one nonzero entry in each row and column. The group of invertible elements of R_n is isomorphic to the symmetric group S_n. Another efficient, combinatorial description of the Bruhat ordering on R_n and a useful, combinatorial formula for the length function on R_n are given by M. Can and L. Renner in [2].

The Bruhat poset of involutions of S_n was first studied by F. Incitti in [4] from a purely combinatorial point of view. He proved that this poset is graded, calculated the rank function and also showed several other important properties of this poset.

In this paper we present a geometric interpretation of this poset and its natural generalization, considering the action of the Borel subgroup on symmetric matrices by congruence. Denote by $\mathbb{B}_n(\mathbb{C})$ the Borel subgroup of $GL_n(\mathbb{C})$, i.e. the group of invertible upper-triangular $n \times n$ matrices over the complex numbers. Denote by $\mathbb{S}(n, \mathbb{C})$ the set of all complex symmetric $n \times n$ matrices. The congruence action of $B \in \mathbb{B}_n(\mathbb{C})$ on $S \in \mathbb{S}(n, \mathbb{C})$ is defined in the following way:

$$S \longmapsto B^t S B .$$

The orbits of this action (to be precisely correct, we must say $S \mapsto \left(B^{-1}\right)^t S B^{-1}$ to get indeed a group action) are called the *congruence B-orbits*. It is known that the orbits of this action may be indexed by partial S_n-involutions (i.e. symmetric $n \times n$ matrices with at most one 1 in each row and in each column) (see [9]).

Thus, if π is such a partial involution, we denote by $\mathcal{C}_\pi$ the corresponding congruence B-orbit of symmetric matrices. The poset of these orbits gives a natural extension of the Bruhat poset of regular (i.e. not partial) involutions of S_n. If we restrict this action to the set of invertible symmetric matrices we get a poset of orbits that is isomorphic to the Bruhat poset of involutions of S_n studied by F. Incitti.

Here, we give another view of the rank function of this poset, combining combinatorics with the geometric nature of it. The rank function equals to the dimension of the orbit variety. We give two combinatorial formulas for the rank function of the poset of partial involutions (Theorems 7.4 and 8.1). The result of Incitti that the Bruhat poset of involutions of S_n is graded and his formula for the rank function of this poset follow from our exposition (Corollary 8.3).

At the end of the paper we briefly discuss how our view of the rank function can be applied to the non-symmetric case, i.e. how to find the rank function of the Bruhat poset of all (not necessarily symmetric) partial permutations in a similar way.

2. Preliminaries

2.1. **Permutations and partial permutations. The Bruhat order.** The Bruhat order on permutations of S_n is defined as follows: $\pi \leqslant \sigma$ if π is is a subword of σ in Coxeter generators $s_1 = (1,2)$, $s_2 = (2,3)$,...,$s_{n-1} = (n-1,n)$. It it well studied from various points of view. The rank function is the length in Coxeter generators which is exactly the number of inversions in a permutation. A permutation matrix is a square matrix which has exactly one 1 in each row and each column while all other entries are zeros. A partial permutation is an injective map defined on a subset of $\{1,2,..,n\}$. A partial permutation matrix is a square matrix which has at most one 1 at each row and each column and all other entries are zeros. So, if we delete the zero rows and columns from a partial permutation matrix we get a (regular) permutation matrix of smaller size, we will use this view later. See works of L. Renner [7] and [8] where the Bruhat order on partial permutations is introduced and studied.

2.2. **Partial order on orbits.** When an algebraic group acts on a set of matrices, the classical partial order on the set of all orbits is defined as follows:

$$\mathcal{O}_1 \leq \mathcal{O}_2 \iff \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$$

where $\overline{S}$ is the (Zarisski) closure of the set S.

Remark 2.1. Note that $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2} \Longrightarrow \overline{\mathcal{O}_1} \subseteq \overline{\mathcal{O}_2}$ for any two sets $\mathcal{O}_1, \mathcal{O}_2$.

3. Rank-control matrices

In this section we define the *rank control matrix* which will turn out to be a key corner in the identification of our poset. We start with the following definition:

Definition 3.1. Let $X = (x_{ij})$ be an $n \times m$ matrix. For each $1 \leq k \leq n$ and $1 \leq l \leq m$, denote by $X_{k\ell}$ the upper-left $k \times \ell$ submatrix of X. We denote by $R(X)$ the $n \times m$ matrix whose entries are: $r_{k\ell} = rank\,(X_{k\ell})$ and call it the *rank control matrix* of X.

It follows from the definitions that for each matrix X, the entries of $R(X)$ are nonnegative integers which do not decrease in rows and columns and each entry is not greater than its row and column number. If X is symmetric, then $R(X)$ is symmetric as well.

Example 3.2.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad R(I_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Remark 3.3. This rank-control matrix is similar to the one introduced by A. Melnikov [5] when she studied the poset (with respect to the covering relation given in Definition 2.2) of adjoint B-orbits of certain nilpotent strictly upper-triangular matrices.

The rank control matrix is connected also to the work of Incitti [4] where regular involutions of S_n are discussed.

Proposition 3.4. *Let $X, Y \in GL_n(\mathbb{F})$ be such that $Y = LXB$ for some invertible lower-triangular matrix L and some Borel (i.e. invertible upper-triangular) matrix B. Denote by $X_{k\ell}$ and $Y_{k\ell}$ the upper-left $k \times \ell$ submatrices of X and Y respectively. Then for all $1 \leqslant k, \ell \leqslant n$*

$$rank\,(X_{k\ell}) = rank\,(Y_{k\ell})\,.$$

Proof.

$$\begin{pmatrix} L_{kk} & 0_{k \times (n-k)} \\ * & * \end{pmatrix} \begin{pmatrix} X_{k\ell} & * \\ * & * \end{pmatrix} \begin{pmatrix} B_{\ell\ell} & * \\ 0_{(n-\ell) \times \ell} & * \end{pmatrix} = \begin{pmatrix} L_{kk} X_{k\ell} B_{\ell\ell} & * \\ * & * \end{pmatrix},$$

and therefore, $Y_{k\ell} = L_{kk} X_{k\ell} B_{\ell\ell}$. The Matrices L_{kk} and $B_{\ell\ell}$ are invertible, which implies that $Y_{k\ell}$ and $X_{k\ell}$ have equal ranks. $\square$

The rank control matrices of two permutations can be used to compare between them in the sense of Bruhat order. This is the reasoning for the next definition:

Definition 3.5. Define the following order on $n \times m$ matrices with positive integer entries: Let $P = (p_{ij})$ and $Q = (q_{ij})$ be two such matrices.

Then

$$P \leqslant_{\mathcal{R}} Q \iff p_{ij} \leqslant q_{ij} \text{ for all } i, j\,.$$

The following lemma appears in another form as Theorem 2.1.5 of [1].

Lemma 3.6. *Denote by $\leqslant_B$ the Bruhat order of S_n and let $\pi, \sigma \in S_n$. Then*

$$\pi \leqslant_{\mathcal{B}} \sigma \iff R(\pi) \geqslant_{\mathcal{R}} R(\sigma)\,.$$

In other words, the Bruhat order on permutations corresponds to the inverse order of their rank-control matrices. $\square$

4. Partial permutations, Partial Involutions and Congruence B-Orbits

Definition 4.1. A partial permutation is an $n \times n$ $(0, 1)$-matrix such that each row and each column contains at most one '1'.

Definition 4.2. If a partial permutation matrix is symmetric, then we call it a *partial involution.*

The following easily verified lemma claims that partial permutations are completely characterized by their rank control matrices.

Lemma 4.3. *For two $n \times n$ partial permutation matrices π, σ we have*

$$R(\pi) = R(\sigma) \iff \pi = \sigma.$$

Remark 4.4. Let U_n be the $n \times n$ upper-triangular matrix with '1's on the main diagonal and in all upper triangle and let π be any partial permutation. Then

$$R(\pi) = U^t \pi U .$$

Theorem 4.5. *There exists a bijection between the set of congruence B-orbits of symmetric matrices over $\mathbb{C}$ and the set of partial involutions.*

Proof. The proof can be obtained by performing a symmetric version of Gauss elimination process. See Theorem 3.2 in [9] for more details. □

Remark 4.6. Note that the elimination process described above works only over $\mathbb{C}$. In the case of $\mathbb{R}$, the diagonal entries of a partial involution matrix (which are its fixed points) belong to $\{0, 1 - 1\}$.

5. The Poset of Congruence B-Orbits of Symmetric Matrices

Here is a direct consequence of Lemma 4.3 and Proposition 3.4.

Proposition 5.1. *All the matrices of a fixed congruence B-Orbit share a comon rank-control matrix. In other words, if π is a partial S_n-involution, and C_π is the congruence B-orbit of symmetric matrices associated with π then*

$$\mathcal{C}_\pi = \{S \in \mathbb{S}(n, \mathbb{C}) \,|\, R(S) = R(\pi)\} .$$

The following lemma describes the orbits:

Lemma 5.2. *Let π be a partial involution and let $R(\pi)$ be its rank-control matrix. Then*

$$\overline{\mathcal{C}_\pi} = \{S \in \mathbb{S}(n, \mathbb{C}) \mid R(S) \leqslant_{\mathcal{R}} R(\pi)\} .$$

Proof. This lemma follows from Theorem 15.31 of [6]. Their exposition differs somewhat from ours as it deals with rectangular, not necessarily symmetric matrices but the differences can be easily overwhelmed by considering also equations of the form $a_{ij} = a_{ji}$ which are polynomial equations with regard to the entries of a matrix. □

Remark 5.3. Over the fields $\mathbb{C}$ and $\mathbb{R}$ the closure in Lemma 5.2 may also be considered with respect to the metric topology.

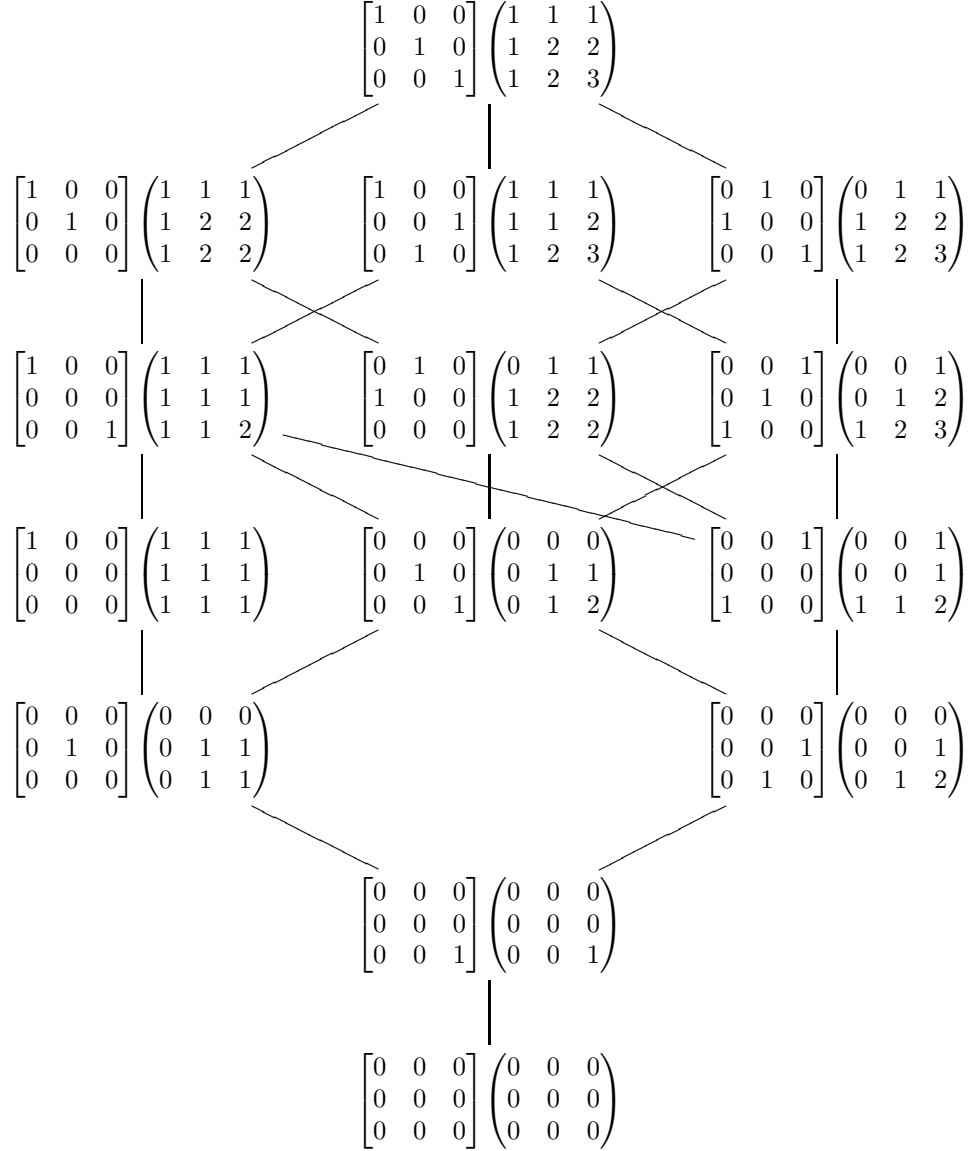The next corollary follows from Lemma 5.2 and characterizes the order relation of the poset of B-orbits.

Corollary 5.4. *Let π and σ be partial S_n-involutions. Then*

$$\mathcal{C}_\pi \leqslant_{\mathcal{O}} \mathcal{C}_\sigma \iff R(\pi) \leqslant_{\mathcal{R}} R(\sigma)$$

6. An example

In this section we give an example for the poset of B-congruence orbits. We represent each orbit by its partial involution (see Theorem 4.5) and write the rank-control matrix together with each partial involution.

Example 6.1. This example illustrates the case $n = 3$.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}1&1&1\\1&2&2\\1&2&3\end{pmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{pmatrix}1&1&1\\1&2&2\\1&2&2\end{pmatrix} \qquad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{pmatrix}1&1&1\\1&1&2\\1&2&3\end{pmatrix} \qquad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{pmatrix}0&1&1\\1&2&2\\1&2&3\end{pmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&2\end{pmatrix} \qquad \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0&1&1\\1&2&2\\1&2&2\end{pmatrix} \qquad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{pmatrix}0&0&1\\0&1&2\\1&2&3\end{pmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \qquad \begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}0&0&0\\0&1&1\\0&1&2\end{pmatrix} \qquad \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}\begin{pmatrix}0&0&1\\0&0&1\\1&1&2\end{pmatrix}$$

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix} \qquad\qquad \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&2\end{pmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

7. The Rank Function

Definition 7.1. A poset P is called *graded* (or *ranked*) if for every $x, y \in P$, any two maximal chains from x to y have the same length.

Proposition 7.2. *The poset of congruence B-orbits of symmetric matrices (with respect to the order $\leqslant_{\mathcal{O}}$) is a graded poset with the rank function given by the dimension of the closure.*

This proposition is a particular case of the following fact. Let G be a connected, solvable group acting on an irreducible, affine variety X. Suppose that there are a finite number of orbits. Let O be the set of G-orbits on X. For $x, y \in O$ define $x \leqslant y$ if $x \subseteq \overline{y}$. Then O is a graded poset.

This fact is given as an exercise in [8] (exercise 12, page 151) and can be proved using the proof of the theorem appearing of Section 8 of [7]. (Note that in our case the Borel group is solvable, the variety of all symmetric matrices is irreducible as a vector space and the number of orbits is finite since there are only finitely many partial permutation.)

A natural problem is to find an algorithm which calculates the $\dim \overline{\mathcal{C}_\pi}$ from a partial permutation matrix π or from its rank-control matrix $R(\pi)$. Here we present such an algorithm.

Definition 7.3. Let π be a partial permutation matrix and let $R(\pi) = (r_{ij})$ be its rank-control matrix. Add an extra 0 row to $R(\pi)$, pushed one place to the left, i.e. assume that $r_{0k} = 0$ for each $0 \leqslant k < n$.

Denote

$$\mathfrak{D}(\pi) = \# \left\{ (i,j) \mid 1 \leqslant i \leqslant j \leqslant n \quad \text{and} \quad r_{ij} = r_{i-1,j-1} \right\}.$$

There are two examples after the proof of Theorem 7.4 (Examples 7.5 and 7.6) and one more example in the end of the paper (Example 8.2) of calculation of the parameter $\mathfrak{D}(\pi)$.

Theorem 7.4. *Let π be a partial S_n-involution. Then*

$$\dim \overline{\mathcal{C}_\pi} = \frac{n^2 + n}{2} - \mathfrak{D}(\pi).$$

Proof. In this proof we use the notion of variety which corresponds to a fragment of a matrix, we put empty boxes instead of entries that we "cut" from a matrix. Any variety of $n \times n$ matrices is a (Zariski) closed subset of the vector space $\mathbb{C}^{n^2}$ and the variety which corresponds to a fragment of $n \times n$ matrix is the closure of the projection of the big variety on the corresponding subspace of $\mathbb{C}^{n^2}$. (Here the projection is the mapping $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ which "forgets" the last coordinate: $p(x_1, x_2, ..., x_{n-1}, x_n) = (x_1, x_2, ..., x_{n-1})$.) Denote by V^{kn} the variety which is a projection of the certain variety $\overline{\mathcal{C}_\pi}$ of symmetric $n \times n$ matrices and corresponds to

the fragment
$$\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1,k} & \cdots & \cdots & a_{1,n-1} & a_{1,n} \\
a_{12} & a_{22} & \cdots & a_{2,k} & \cdots & \cdots & a_{2,n-1} & a_{2,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1,k-1} & a_{2,k-1} & \cdots & a_{k-1,k} & \cdots & \cdots & a_{k-1,n-1} & a_{k-1,n} \\
a_{1,k} & a_{2,k} & \cdots & a_{k,k} & \cdots & \cdots & a_{k,n-1} & a_{k,n} \\
a_{1,k+1} & a_{2,k+1} & \cdots & a_{k,k+1} & \cdots & \cdots & a_{k+1,n-1} & \square \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \square \\
a_{1,n-1} & a_{2,n-1} & \cdots & & \cdots & \cdots & a_{n-1,n-1} & \square \\
a_{1,n} & a_{2,n} & \cdots & a_{k,n} & \square & \square & \square & \square
\end{bmatrix}.$$
(For V^{kn} the last non empty entry in the n-th column is in the row number k, all further positions in the n-th row and column are empty.) Consider also the variety

$V^{k-1,n}$ which corresponds to
$$
\begin{bmatrix}
a_{11} & \cdots & a_{1,k-1} & \cdots & a_{1,n-1} & a_{1,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1,k-1} & \cdots & a_{k-1,k-1} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\
a_{1,k} & \cdots & a_{k-1,k} & \cdots & a_{k,n-1} & \square \\
\cdots & \cdots & \cdots & \cdots & \cdots & \square \\
a_{1,n-1} & \cdots & & \cdots & a_{n-1,n-1} & \square \\
a_{1,n} & \cdots & a_{k-1,n} & \square & \square & \square
\end{bmatrix}.
$$

Note that since V^{kn} and $V^{k-1,n}$ are projections of the same variety $\overline{\mathcal{C}_\pi}$ and V^{kn} has one more coordinate than $V^{k-1,n}$, there are only two possibilities for their dimensions: $\dim V^{kn} = \dim V^{k-1,n}$ or $\dim V^{kn} = \dim V^{k-1,n} + 1$.

Now, let us start the course of the proof, by induction on n. For $n = 1$ the statement is obviously true.

Consider an $n \times n$ partial S_n involution π_n and its rank-control matrix $R(\pi_n)$. Its upper-left $(n-1) \times (n-1)$ submatrix is the rank-control matrix $R(\pi_{n-1})$ of the partial S_{n-1}-involution π_{n-1} which is the upper-left $(n-1) \times (n-1)$ submatrix of the partial S_n-involution π. By the induction hypothesis, $\dim \overline{\mathcal{C}_{\pi_{n-1}}} = \frac{n^2-n}{2} - \mathfrak{D}(\pi_{n-1})$. Now we add the n-th column to the partial involution matrix (the n-th row is the same and does not provide any new information because we deal with the symmetric matrices) and consider the n-th column of $R(\pi)$. (We also add the n-th row but since our matrices are symmetric it suffices to understand only what happens to the dimension when we add the n-th column.) We added n new coordinates to the variety $\overline{\mathcal{C}_{\pi_{n-1}}}$ and we have to show that

$$
\dim \overline{\mathcal{C}_\pi} = \dim \overline{\mathcal{C}_{\pi_{n-1}}} + n - \#\{(i,n) \,|\, 1 \leqslant i \leqslant n \quad \text{and} \quad r_{in} = r_{i-1,n-1}\}, \qquad (*)
$$

i.e. not all the n coordinates that we added make the dimension greater but only those of them for which there is an inequality in the corresponding place of the certain diagonal of the rank-control matrix. The equality $(*)$ implies the statement of our theorem since $\frac{n^2-n}{2} + n = \frac{n^2+n}{2}$ and

$$
\mathfrak{D}(\pi) = \mathfrak{D}(\pi_{n-1}) + \#\{(i,n) \,|\, 1 \leqslant i \leqslant n \quad \text{and} \quad r_{in} = r_{i-1,n-1}\}.
$$

Obviously, if $r_{1,n} = 0$, then $a_{1,n} = 0$ for any $A = (a_{ij})_{i,j=1}^n \in \overline{\mathcal{C}_\pi}$, this itself is a polynomial equation which makes the dimension lower by 1, while if $r_{1,n} = 1$ it means that the rank of the first row is maximal and therefore, no equation. (In other words, the dimension of the variety V^{1n} which corresponds

to
$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{12} & a_{22} & \cdots & a_{2,n-1} & \square \\
\cdots & \cdots & \cdots & \cdots & \square \\
a_{1,n-1} & a_{2,n-1} & \cdots & a_{n-1,n-1} & \square \\
a_{1,n} & \square & \square & \square & \square
\end{bmatrix}
$$
is greater by one than the of the va-

riety V^{0n} which corresponds to
$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1,n-1} \\
a_{21} & a_{22} & \cdots & a_{2,n-1} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1}
\end{bmatrix}
$$
when $r_{1,n} = 1$

and they have equal dimensions when $r_{1,n} = 0$.) Now move down along the n-th column of $R(\pi)$. Again by induction, this time the induction is on the number of row k, assume that for each $1 \leqslant i \leqslant k-1$ the dimensions of V^{in} and $V^{i-1,n}$ ar equal iff $r_{i-1,n-1} = r_{i,n}$ and $\dim V^{in} = \dim V^{i-1,n} + 1$ iff $r_{i-1,n-1} < r_{i,n}$. First,

let $r_{k-1,n-1} = r_{k,n} = c$. Consider a matrix $A = (a_{ij})_{i,j=1}^{n} \in \overline{\mathcal{C}_\pi}$ and consider its

upper-left $(k-1) \times (n-1)$ submatrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,n-1} \end{bmatrix}$. Using the

notation introduced in Proposition 3.4, we denote this submatrix as $A_{k-1,n-1}$. If $c = 0$, then $rank A_{kn} = 0$, so A_{kn} is a zero matrix and $\dim V^{in} = \dim V^{i-1,n} = 0$.

Let $c \neq 0$. Since $rank(A_{k-1,n-1}) = c$, we can take c linearly independent columns

$\begin{bmatrix} a_{1,j_1} \\ a_{2,j_1} \\ \cdots \\ a_{k-1,j_1} \end{bmatrix}, \ \ldots \ , \ \begin{bmatrix} a_{1,j_c} \\ a_{2,j_c} \\ \cdots \\ a_{k-1,j_c} \end{bmatrix}$ which span its column space. Now take only linearly

independent rows of the $(k-1) \times c$ matrix $\begin{bmatrix} a_{1,j_1} & \cdots & a_{1,j_c} \\ a_{2,j_1} & \cdots & a_{2,j_c} \\ \cdots & \cdots & \cdots \\ a_{k-1,j_1} & \cdots & a_{k-1,j_c} \end{bmatrix}$ to get a nonsin-

gular $c \times c$ matrix $\begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_c} \\ a_{i_2,j_1} & \cdots & a_{i_2,j_c} \\ \cdots & \cdots & \cdots \\ a_{i_c,j_1} & \cdots & a_{i_c,j_c} \end{bmatrix}$. The equality $r_{k-1,n-1} = r_{k,n} = c \leqslant k - 1$

implies that any $(c+1) \times (c+1)$ minor of the matrix A_{kn} is zero, in particular

$\det \begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_c} & a_{i_1,n} \\ a_{i_2,j_1} & \cdots & a_{i_2,j_c} & a_{i_2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_c,j_1} & \cdots & a_{i_c,j_c} & a_{i_c,n} \\ a_{k,j_1} & \cdots & a_{k,j_c} & a_{k,n} \end{bmatrix} = 0$, which is a polynomial equation. This equa-

tion is algebraically independent of the similar equations obtained for $1 \leqslant i \leqslant k - 1$ because it involves the "new" variable – the entry $a_{k,n}$. It indeed involves the entry

$a_{k,n}$ since $\det \begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_c} \\ a_{i_2,j_1} & \cdots & a_{i_2,j_c} \\ \cdots & \cdots & \cdots \\ a_{i_c,j_1} & \cdots & a_{i_c,j_c} \end{bmatrix} \neq 0$. This equation means that the variable

$a_{k,n}$ is not independent of the coordinates of the variety $V^{k-1,n}$, and therefore $\dim V^{k-1,n} = \dim V^{kn}$.

Now let $r_{k-1,n-1} < r_{k,n} = c$, and we have to show that in this case the variable a_{nk} is independent of the coordinates of $V^{k-1,n}$, in other words, we have to show that there is no new equation. Consider the fragment $\begin{bmatrix} r_{k-1,n-1} & r_{k-1,n} \\ r_{k-1,n} & r_{k,n} \end{bmatrix}$. There are four possible cases:

$$\begin{bmatrix} r_{k-1,n-1} & r_{k-1,n} \\ r_{k-1,n} & r_{k,n} \end{bmatrix} = \begin{bmatrix} c-1 & c-1 \\ c-1 & c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c-2 & c-1 \\ c-1 & c \end{bmatrix} \quad \text{or}$$
$$\begin{bmatrix} c-1 & c \\ c-1 & c \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c-1 & c-1 \\ c & c \end{bmatrix}.$$

The equality $r_{k,n} = c$ implies that each $(c+1) \times (c+1)$ minor of A_{kn} is equal to zero, but we shall see that each such equation is not new, i.e. it is implied by the equality $r_{k,n-1} = c - 1$ or by the equality $r_{k-1,n} = c - 1$. In the first three of above four

cases we decompose the $(c+1) \times (c+1)$ determinant $\det \begin{bmatrix} \cdots & \cdots \\ \cdots & a_{k,n} \end{bmatrix}$ using the last column. Since in all these cases $r_{k,n-1} = c-1$, each $c \times c$ minor of this decomposition (i.e. each $c \times c$ minor of $A_{k,n-1}$) is zero and therefore, this determinant is zero. In the fourth case we get the same if we decompose the determinant using its last row instead of the last column: since $r_{k-1,n} = c - 1$, all the $c \times c$ minors of this decomposition (i.e. all $c \times c$ minor of $A_{k-1,n}$) are zeros and thus, our $(c+1) \times (c+1)$ determinant equals to zero. So there is no algebraic dependence between a_{kn} and the coordinates of $V^{k-1,n}$. Therefore, $\dim V^{kn} = \dim V^{k-1,n} + 1$. The case $k = n$ is the same as other cases when $k \leqslant n - 1$. The proof is completed. $\square$

We end this section with two examples:

Example 7.5. If $\pi = Id$ then clearly $\mathfrak{D}(\pi) = 0$ and $\dim \overline{\mathcal{C}_\pi} = \frac{n^2+n}{2}$. Indeed, $\overline{\mathcal{C}_{Id}}$ is the variety of all symmetric $n \times n$ matrices and thus its dimension is equal to n^2 minus the number of equations of the type $a_{ij} = a_{ji}$, i.e. $\dim \overline{\mathcal{C}_{Id}} = n^2 - \frac{n^2-n}{2} = \frac{n^2+n}{2}$.

Example 7.6. Taking $\pi = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $R(\pi) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ we have diagonals (with added zeros) $(0, 0, 2, 3)$, $(0, 1, 2)$ and $(0, 1)$. We see that here we have one place in the beginning of the main diagonal where $r_{11} = r_{00} = 0$ while in all other places r_{ij} is strictly greater than $r_{i-1,j-1}$. Therefore, $\mathfrak{D}(\pi) = 1$ and $\dim \overline{\mathcal{C}_\pi} = \frac{3^2+3}{2} - 1 = 5$. Indeed,

$$\overline{\mathcal{C}_\pi} = \left\{ (a_{ij})_{i,j=1}^3 \mid a_{12} = a_{21}, a_{13} = a_{31}, a_{23} = a_{32} \text{ and } a_{11} = 0 \right\}.$$

The dimension of the vector space of all 3×3 matrices is $3^2 = 9$ and here we have four algebraically independent equations, so the dimension is 5.

8. Another characterization of the parameter $\mathfrak{D}(\pi)$

Obviously, an $n \times n$ partial involution matrix π can be described uniquely by the pair $(\tilde{\pi}, \{i_1, ..., i_k\})$, where $n - k$ is the rank of the matrix π, $\tilde{\pi} \in S_{n-k}$ such that $\tilde{\pi}^2 = Id$ is the regular (not partial) involution of the symmetric group S_{n-k} and the integers $i_1, ..., i_k$ are the numbers of zero rows (columns) in the matrix π.

The following theorem is a generalization of the formula for the rank function of the Bruhat poset of the involutions of S_n given by Incitti in [4]. It is indeed the rank function because we already know that the rank function is the dimension (Proposition 7.2) and the dimension is determined by the parameter $\mathfrak{D}$ (Theorem 7.4).

Theorem 8.1. *Following Incitti, denote by $Invol(G)$ the set of all involutions in the group G. Then for a partial permutation $\pi = (\tilde{\pi}, \{i_1, ..., i_k\})$, where $\tilde{\pi} \in Invol(S_{n-k})$ and the integers $i_1, ..., i_k$ are the numbers of zero rows (columns) in the matrix π is:*

$$\mathfrak{D}(\pi) = \frac{exc(\tilde{\pi}) + inv(\tilde{\pi})}{2} + \sum_{t=1}^{k} (n + 1 - i_t)$$

In other words, $\mathfrak{D}(\pi)$ equals to the length of $\tilde{\pi}$ in the poset of the involutions of the group S_{n-k} plus the sum of the numbers of zero rows of the matrix π, where

the numbers are taken in the opposite order, i.e. the n-th row is labeled by 1, the $(n-1)$-th row is labeled by 2,..., the first row is labeled by n.

Proof. We prove by induction on n. The claim is trivial for $n = 1$ so assume that it is true for $n-1$ and let π_n be a partial permutation of order n. Let π_{n-1} be the sub matrix of π_n consisting of the first $n-1$ rows and columns. We introduce some notations:

- Let $\pi_n(n) = i_1$. This means that the digit 1 of column n appears at row i_1. If $i_1 = 0$ then column n is a zero column.
- Denote by O_n, (O_{n-1}) the set of zero rows of π_n, (π_{n-1}) respectively.
- Let π_k be a partial permutation of order k. For each $1 \leq i \leq k$, the number of zero columns of π_k, until column i (including i itself in case it is a zero column) is denoted by $o_k(i)$.
- Let

$$\Delta_n = inv(\tilde{\pi}_n) + exc(\tilde{\pi}_n) - (inv(\tilde{\pi}_{n-1}) + exc(\tilde{\pi}_{n-1})).$$

- If π is a partial permutation on n elements then we denote

$$sup(\pi) = |\{1 \leq i \leq n \mid \pi(i) \neq 0\}\}|$$

so $\tilde{\pi} \in S_{\sup(\pi)}$.
- For each partial involution π of order k and $1 \leq i \leq k$ such that $\pi(i) \neq 0$ one has: $\pi(i) = j$ if and only if $\tilde{\pi}(i - o_k(i)) = j - o_k(j)$.
- For a partial involution π

$$exc(\tilde{\pi}) = \frac{sup(\pi) - fix(\pi)}{2}$$

where $fix(\pi) = |\{1 \leq i \leq sup(\pi) \mid \pi(i) = i\}|$.

It is sufficient to prove that

$$\mathfrak{D}(\pi_n) - \mathfrak{D}(\pi_{n-1}) = \frac{\Delta_n}{2} + \sum_{i \in O_n} (n + 1 - i) - \sum_{i \in O_{n-1}} (n - i)$$

We calculate first the L.H.S.
Recall that

$$\mathfrak{D}(\pi_n) - \mathfrak{D}(\pi_{n-1}) = \{i \mid 1 \leq i \leq n, r_{i_n} = r_{i-1,n-1}\}.$$

Let $R(\pi_n) = (r_{k,l})_{1 \leq k,l \leq n}$ be the rank control matrix of π_n and let $i \in \{1, \ldots, n-1\}$. If $i \notin O_{n-1}$ then $r_{i,n} \neq r_{i-1,n-1}$. If $i \in O_{n-1} - O_n$ then $\pi_n(n) = i$ and we have again $r_{i,n} \neq r_{i-1,n-1}$. If $i \in O_{n-1} \cap O_n$ then $r_{i,n} = r_{i-1,n-1}$ if and only if the digit 1 of column n of π_n appears after the row i or if is does not appear at all, i.e. $\pi_n(n) > i$ or $\pi_n(n) = 0$. Thus

$$\mathfrak{D}(\pi_n) - \mathfrak{D}(\pi_{n-1}) = \begin{cases} |\{i \mid \pi_n(n) > i\} \cap O_n| = o_n(i_1) & i_1 \neq 0 \\ |O_n| & i_1 = 0 \end{cases}$$

Before calculating the R.H.S., note that if $\pi \in S_k$ is an involution then

$$exc(\pi) = \frac{k - fix(\pi)}{2}$$

where $fix(\pi) = |\{1 \leq i \leq k \mid \pi(i) = i\}|$. We distinguish between three cases according to the value of $i_1 = \pi_n(n)$:

(1) $i_1 = 0$

In this case we have $\tilde{\pi}_n = \tilde{\pi}_{n-1}$ so $\Delta_n = 0$. We also have $|O_n| = |O_{n-1}| + 1$ so that the R.H.S is just $|O_n|$ as required.

(2) $0 < i_1 < n$. Note first that in this case $fix(\tilde{\pi}_{n-1}) = fix(\tilde{\pi}_n)$. (Indeed, let $k \in \{n-1, n\}$ and let $i \in \{1, \ldots, n\}$ be such that $\pi_k(i) = j \neq 0$. Then $i - o_k(i)$ is a fixed point of $\tilde{\pi}_k$ if and only if $i - o_k(\mathfrak{1}) = j - o_k(j)$. If (without loss of generality) $i < j$ then the number of zero columns between i and j is equal to the total number of columns between i and j in π_k which implies that $j = 0$, a contradiction. Thus we must have $i = j$ so $i - o_k(i)$ is a fixed point of $\tilde{\pi}_k$ if and only if i is a fixed point of π_k. The only difference between π_n is at i_1 which can't be a fixed point, hence $fix(\tilde{\pi}_{n-1}) = fix(\tilde{\pi}_n)$).

It is easy to see that $sup(\pi_n) = sup(\pi_{n-1}) + 2$ and thus $exc(\pi_n) - exc(\pi_{n-1}) = 1$.

We turn now to the calculation of $inv(\tilde{\pi}_n) - inv(\tilde{\pi}_{n-1})$. When we pass from $\tilde{\pi}_{n-1}$ to $\tilde{\pi}_n$, we put 1 in places $(n - o_n(n), i_1 - o_n(i_1))$ and $(i_1 - o_n(i_1), n - o_n(n))$. The columns after the column $i_1 - o_n(i_1)$ contribute $n - o_n(n) - (i_1 - o_n(i_1)) = n - o_n(n) - i_1 + o_n(i_1)$ inversions while the rows after row $i_1 - o_n(i_1)$ contribute $n - o_n(n) - (i_1 - o_n(i_1))$ inversions. Since the 1 in place $(n - o_n(n), i_1 - o_n(i_1))$ was counted twice, we have:

$$inv(\tilde{\pi}_n) - inv(\tilde{\pi}_{n-1}) = 2(n - o_n(n) - i_1 + o_n(i_1)) - 1.$$

Now, we have:

$$\frac{\Delta_n}{2} = \frac{exc(\tilde{\pi}_n) - exc(\tilde{\pi}_{n-1}) + inv(\pi_n) - inv(\pi_{n-1})}{2} =$$
$$= \frac{1 + 2(n - o_n(n) - i_1 + o_n(i_1)) - 1}{2} =$$
$$= n - o_n(n) - i_1 + o_n(i_1) = n - |O_n| - i_1 + o_n(i_1).$$

Now, since $0 < i_1 < n$, we have $|O_n| = |I_{n-1}|$, $\sum_{i \in I_{n-1}} i - \sum_{i \in O_n} i = i_1$ and we are done.

(3) $i_1 = n$: The following facts are easy to verify in this case:

$$inv(\tilde{\pi}_n) - inv(\tilde{\pi}_{n-1}) = exc(\tilde{\pi}_n) - exc(\tilde{\pi}_{n-1}),$$

$$|O_n| = |O_{n-1}| = |o_n(n)| = |o_n(i_1)|,$$

and from here we easily get our result. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 8.2.

$$\pi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R(\pi) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 2 & 3 & 4 & 4 \end{pmatrix}$$

The diagonals with added zeros are $(0, 0, 0, 0, 2, 4, 4)$, $(0, 0, 0, 1, 3, 4)$, $(0, 0, 1, 2, 3)$, $(0, 1, 2, 2)$, $(0, 1, 2)$ and $(0, 1)$. Thus, $\mathfrak{D}(\pi) = 8$.

From the other hand, here we have $n = 6$,

$$\tilde{\pi} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = (3412) \in S_4 \text{ , and zero rows of } \pi \text{ are } \{i_1, i_2\} = \{3, 6\}\,.$$

So,

$$\mathfrak{D}(\pi) = \frac{exc(\tilde{\pi}) + inv(\tilde{\pi})}{2} + \sum_{t=1}^{2}(6 + 1 - i_t) = \frac{2 + 4}{2} + 4 + 1 = 8\,.$$

Corollary 8.3. *The Bruhat poset of regular (not partial) involutions of S_n is a graded poset with the rank function given by the formula*

$$\mathfrak{D}(\sigma) = \frac{exc(\sigma) + inv(\sigma)}{2}\,,$$

where $\sigma \in Invol(S_n)$.

Proof. The fact that the Bruhat poset of involutions of S_n is graded follows from the fact that this poset is an interval (more precisely, a reversed interval) in the poset of partial involutions (since the (n, n)-th entry in the rank-control matrix of any regular involution is n because the matrix of $\sigma \in Invol(S_n)$ is invertible) and the fact that the poset of partial involutions is graded (Proposition 7.2). The formula for the rank function follows from Theorem 8.1 since the matrix of $\sigma \in Invol(S_n)$ is invertible and doesn't have zero rows or columns. This is exactly the rank function of the Bruhat poset of the involutions of S_n given by Incitti in [4]. $\square$

9. THE NON-SYMMETRIC CASE

Consider the the double cosets $\mathcal{B}_\pi = \{B_1^t \pi B_2\}$ where $\pi \in R_n$ (i.e. π is a partial permutation and $B_1, B_2 \in \mathbb{B}_n(\mathbb{C})$. (Considering only $\pi \in S_n$ we get a version of the Bruhat decomposition for $GL_n(\mathbb{C})$). Similarly to the definition of $\mathfrak{D}$ (Definition 7.3) we can define a parameter $\mathfrak{E}(\pi)$ for a not necessarily symmetric π which counts all the equalities in all the diagonals of the rank-control matrix of π, not only in its upper triangle as we did for the symmetric case. Then we have the following formula for the dimension:

$$\dim \overline{\mathcal{B}_\pi} = n^2 - \mathfrak{E}(\pi)\,.$$

Comparing this formula with the formula of Theorem 7.4 we see that here we have n^2 instead of $\frac{n^2+n}{2}$ because the dimension of the variety of all $n \times n$ matrices is n^2. The proof is same as he proof of Theorem 7.4 with obvious changes. Another case where the similar to $\mathfrak{D}$ parameter is used in the similar way is discussed in [3], where the poset of Borel congruence classes of anti-symmetric matrices is presented. In [3] it is shown that congruence B-orbits of anti-symmetric matrices can be indexed by involutions of the symmetric group and the parameter which is analogous to $\mathfrak{D}$ (denoted as $\mathfrak{A}$ in [3]) to counts the equalities in the diagonals of the "strict" upper triangle of the rank-control matrix (i.e the upper triangle without the main diagonal). There is the a formula (Theorem 5.6 of [3]) for the dimension of the closure of the congruence B-orbit $\mathcal{A}_\pi$ which corresponds to the involution $\pi \in S_n$:

$$\dim \overline{\mathcal{A}_\pi} = \frac{n^2 - n}{2} - \mathfrak{A}(\pi).$$

The number $\frac{n^2-n}{2}$ is the dimension of the variety of all $n \times n$ anti-symmetric matrices.

So, the counting of the equalities in the diagonals of the rank-control matrix is a unified way to find the rank function of the (generalized) Bruhat poset, both in the general and in the symmetric case and also for the poset of congruence B-orbits of anti-symmetric matrices, i.e. the same must be done to compute the rank function of the Bruhat poset all (partial) permutations and the Bruhat poset (partial) involutions. The difference between these three cases is very natural: in the general case we consider all the diagonals of the rank-control matrix, while in the symmetric and anti-symmetric case we consider only the upper triangle (with or without the main diagonal respectively), since a symmetric matrix is completely determined by its upper triangle, and an anti-symmetric matrix has zeros in its main diagonal and is completely determined by its "strict" upper triangle.

If we consider the double cosets $\{B_1 \pi B_2\}$ (multiplying π from both sides by upper-triangular matrices as L. Renner does in his works [8], [7] and [2]) we also can define a parameter analogous to $\mathfrak{E}$ or $\mathfrak{D}$, but in this case we must go along the diagonals from south-west to the north-east (from the lower-left corner to the upper-right corner) of the rank-control matrix. In this case case the number of such equalities is equal to the dimension. The proof also is similar to the proof of Theorem 7.4.

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